On rings with finite number of orbits

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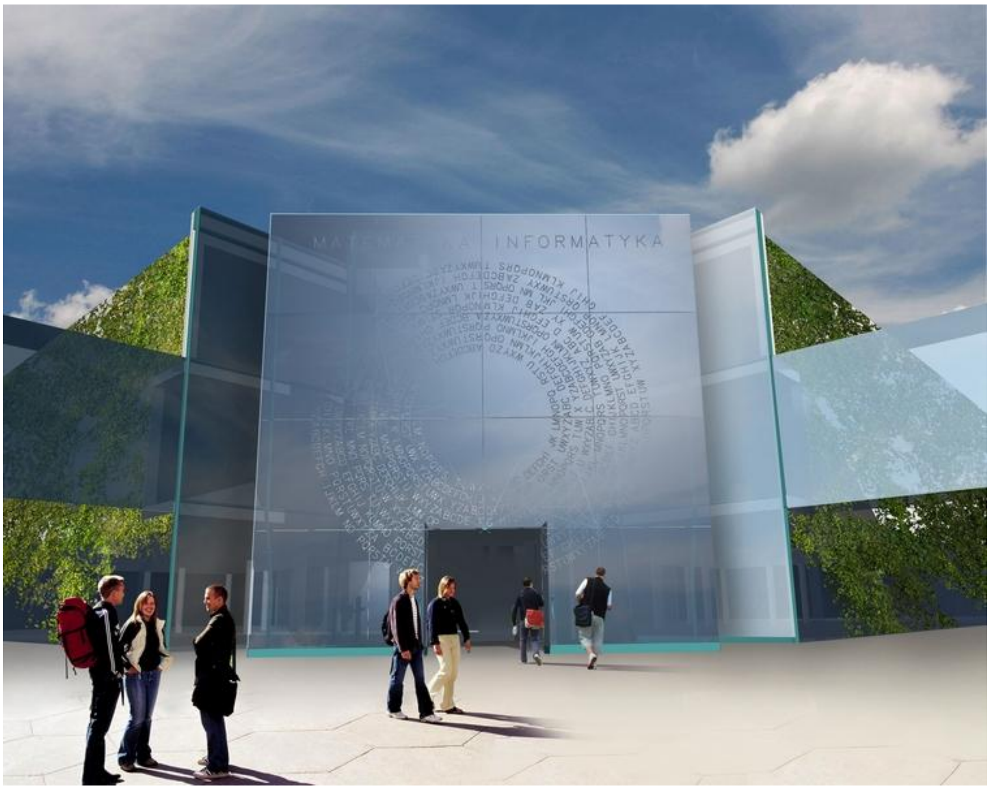

A ring means an associative ring with unit. For a ring R, R^+ denotes the additive group of R. The unit group of R is denoted by $U(R)$. The Jacobson radical of R is denoted by $J(R)$.

Yasuyuki Hirano in [*Rings with finitely many orbits under the regular action,* Lecture Notes in Pure and Appl. Math. 236, Dekker, New York 2004, 343–347] concentrated on the left regular group action of $U(R)$ on R^+ defined by

$$a \rightharpoonup x = ax$$

for all $a \in U(R)$, $x \in R$.

Theorem (Hirano). For a ring R, the following conditions are equivalent:

1. R has only a finite number of orbits under the left regular group action of $U(R)$ on R^+.
2. R has only a finite number of left ideals.
3. R is the direct sum of a finite ring and a finite number of left uniserial rings (that is, rings which left ideals form a finite chain).

If any of these conditions holds, then R is a left artinian ring. More precisely, R is the direct sum of a finite ring and a finite number of principal left ideal left artinian rings. $\square$

We will see later that a ring satisfying conditions of the Hirano Theorem need not be right artinian.

We concentrate on the two-sided regular group action of $U(R) \times U(R)$ on R^+ defined by

$$(a, b) \rightharpoonup x = axb^{-1}, \tag{1}$$

for all $a, b \in U(R)$, $x \in R$.

The action (1) induces an action of the group $U(R) \times U(R)$ on each of the following sets: the set of elements of R, of principal left ideals of R, of left ideals of R, and of ideals of R, however the action on the latter set is trivial. Orbits under the action (1) are called simply U-orbits.

We introduce the following properties:

FNE R has only a finite number of U-orbits of elements.

FNPLI R has only a finite number of U-orbits of principal left ideals.

FNLI R has only a finite number of U-orbits of left ideals.

FNI R has only a finite number of U-orbits of ideals (R has only a finite number of ideals).

For a ring R, the following connections between the above properties holds:

$$\begin{aligned} & FNE \Rightarrow FNPLI \Rightarrow FNI \\ \text{and} \quad & FNLI \Rightarrow FNPLI \Rightarrow FNI \end{aligned} \tag{2}$$

Theorem. For a commutative ring R, the following statements are equivalent:

1. R satisfies each of the properties listed in Formula (2).

2. R satisfies any of the properties listed in Formula (2).

3. R is the direct sum of a finite ring and a finite number of principal ideal local artinian rings. $\square$

We will see later that in non-commutative case the converse of the implications listed in Formula (2) is not necessarily true. Although, accordingly to [Jan Okniński, Lex E. Renner, *Algebras with finitely many orbits,* J. Algebra 264 (2003), 479–495], under the assumption on semiperfectness* of a ring, the property FNPLI implies the property FNE.

*A ring R is called semiperfect if R is semilocal (that is, $R/J(R)$ is semisimple artinian), and idempotents of $R/J(R)$ can be lifted to R.

We discuss two questions:

* Must every left and/or right artinian ring satisfy FNE or a similar property?

* Must every ring satisfying FNE or a similar property be left and/or right artinian, or at least semiprimary*?

Theorem. Every semisimple artinian ring satisfies all the properties listed in Formula (2). □

*A ring R is called semiprimary if R is semilocal, and $J(R)$ is nilpotent.

Example. Let $\mathbb{K}$ be an infinite field, and let $R = \mathbb{K}[x, y]/(x^2, xy, y^2)$ be the homomorphic image of the polynomial ring in commuting variables x, y. Then

- R is a 3-dimensional $\mathbb{K}$-algebra.

- R has an infinite number of U-orbits of ideals.

$\square$

- Under which conditions does a left and/or right artinian ring satisfy FNE or a similar property?

Theorem. Assume a ring R satisfies FNI. Then

1. $P(R)$ is nilpotent, where $P(R)$ denotes the prime radical of R.

2. If R is left or right noetherian, then $J(R)$ is nilpotent.

3. If every prime image of R is simple artinian, then R is semiprimary.

4. If R satisfies a polynomial identity, then R is semiprimary.

Proof. The statement 1 follows from the definition of the prime radical of R as the sum of a finite number of nilpotent ideals.

The statement 2 follows from the Nakayama Lemma.

3. Let $P_1, P_2, \ldots, P_n$ be all prime ideals of R. By assumption, the prime images $R/P_1, R/P_2, \ldots, R/P_n$ of R are simple artinian, and hence $P_1, P_2, \ldots, P_n$ are all maximal ideals of R. According to the Chinese Remainder Theorem for rings, $R/P(R) \cong R/P_1 \times R/P_2 \times \ldots \times R/P_n$ is a semisimple artinian ring. Thus $J(R) = P(R)$ is nilpotent, and $R/J(R)$ is semisimple artinian.

4. If R is a prime PI-ring satisfying FNI, then R is a central prime PI-algebra, and according to the Kaplansky Theorem, R is a simple artinian ring. Let R be now an arbitrary PI-ring satisfying FNI. From the statement 3, R is a semiprimary ring. $\square$

Lemma. Assume a ring R satisfies FNPLI. Then every one-sided nil-ideal of R is nilpotent of nilpotency index not greater than $n+1$, where n denotes the number of U-orbits of principal left ideals of R.

Proof. Let I be a one-sided nil-ideal of R. Suppose that $x_1 x_2 \cdots x_{n+1} \neq 0$ for some $x_1, x_2, \ldots, x_{n+1} \in I$. Out of all the left ideals $Rx_1, Rx_1x_2, \ldots, Rx_1x_2 \cdots x_{n+1}$ of R at least two belong to the same U-orbit, say $Rx_1x_2 \cdots x_i$ and $Rx_1x_2 \cdots x_i x_{i+1} \cdots x_j$. Set $x = x_1x_2 \cdots x_i$ and $y = x_{i+1} \cdots x_j$. Then $x = rxyb^{-1}$ for some $r \in R$, $b \in U(R)$. By induction on $m \geq 1$, $x = r^m x(yb^{-1})^m$. But $(yb^{-1})^m = 0$ for any sufficiently large m, which contradicts $x \neq 0$. □

Theorem. Assume a ring R satisfies FNPLI. Then R is semiprimary provided at least one of the following conditions is fulfilled:

1. R is semilocal, and $J(R)$ is nil.

2. R satisfies ACC or DCC on principal left ideals.

Proof. The statement 1 follows from the previous lemma.

2. If R does not satisfy DCC on principal left ideals, then there exist left ideals $Rx \subsetneq Ry$ of R belonging to the same U-orbit, hence $Rxb^{-1} = Ry$ for some $b \in U(R)$, and thus $Rx \subsetneq Rxb^{-1} \subsetneq Rxb^{-2} \subsetneq \ldots$, which means that R does not satisfy ACC on principal left ideals, contrary to assumption. Thereby R must satisfy DCC on principal left ideals. According to the Bass Theorem, R is right perfect*. In particular, R is semilocal, and $J(R)$ is nil. From the statement 1, R is a semiprimary ring. □

*A ring R is called right perfect if R is semilocal and $J(R)$ is right T-nilpotent.

Jan Okniński and Lex E. Renner in [*Algebras with finitely many orbits,* J. Algebra 264 (2003), 479–495] conjectured that every ring satisfying FNLI is semiprimary.

Theorem. Assume a ring R satisfies FNLI. Then R is semilocal. Moreover, R is semiprimary provided at least one of the following conditions is fulfilled:

1. $J(R)$ is nil.

2. Every prime image of R is left bounded* (such a ring R is called left fully bounded).

*A ring R is called left bounded if every essential left ideal of R contains a non-zero ideal of R.

Proof. The semilocalness of R was in fact proved by Jan Okniński and Lex E. Renner. Suppose that there exists a strictly decreasing sequence of left ideals $M_1 \supsetneq M_1 \cap M_2 \supsetneq M_1 \cap M_2 \cap M_3 \supsetneq \ldots$ for some maximal left ideals $M_1, M_2, \ldots$ of R. From $(M_1 \cap M_2 \cap \ldots \cap M_k)/(M_1 \cap M_2 \cap \ldots \cap M_{k+1}) \cong R/M_{k+1}$ we see that $R/(M_1 \cap M_2 \cap \ldots \cap M_n)$ is a left R-module of length n, for every $n \geq 1$. On the other hand, there exist $m \neq n$ for which left ideals $M_1 \cap M_2 \cap \ldots \cap M_m$ and $M_1 \cap M_2 \cap \ldots \cap M_n$ belong to the same U-orbit, and hence $R/(M_1 \cap M_2 \cap \ldots \cap M_m) \cong R/(M_1 \cap M_2 \cap \ldots \cap M_n)$ as left R-modules, contrary to the Jordan-Hölder Theorem. Thereby $J(R) = M_1 \cap M_2 \cap \ldots \cap M_n$ for some $n \geq 1$, and thus $R/J(R)$ is a semisimple artinian ring.

The statement 1 follows from the previous lemma.

2. Let R be a prime left bounded ring satisfying FNLI. Let I be a minimal ideal of R, and let $0 \neq a \in I$. Let M be a left ideal of R maximal with respect to $a \notin M$. Since every non-zero submodule of R/M contains $a+M$, it follows that $(M+Ra)/M$ is a simple left R-module. On the other hand, by assumption, M is a non-essential left ideal of R, hence $M \subsetneq M \oplus N$ for some left ideal N of R, and by maximality of M, $M + Ra \subseteq M \oplus N$. Thus $(M + Ra)/M \subseteq (M \oplus N)/M \cong N \subseteq R$, which means that R contains simple submodules (minimal left ideals), say $L_1, L_2, \ldots$. Out of all the left ideals $L_1, L_1 \oplus L_2, \ldots$ of R none two of them belong to the same U-orbit. Thereby R has only a finite number of minimal left ideals, and in consequence, is a simple artinian ring. Let R be now an arbitrary left fully bounded ring satisfying FNLI. From one of the previous theorem, R is a semiprimary ring. □

Theorem. Let R be a semiprimary ring. Then R is left artinian provided at least one of the following conditions is fulfilled:

1. R satisfies FNLI.

2. $J(R)$ is a finitely generated R-module.

3. R satisfies FNI, and is a finitely generated PI-algebra over its center.

In particular, every ring satisfying both FNLI and ACC or DCC on principal left ideals is left artinian.

$\square$

Example. Let $\mathbb{L}$ be a field, let $\mathbb{F} = \mathbb{L}(x)$ be the field of rational functions in one variable x, and let $\sigma\colon \mathbb{F} \to \mathbb{F}$ be the $\mathbb{L}$-endomorphism defined by $\sigma(x) = x^n$ for some positive integer $n \geq 2$. Let $R = \mathbb{F}[y;\sigma]/(y^2)$ be the homomorphic image of the skew polynomial ring in one variable y. Then

$*$ R is both left and right artinian.

$*$ R has exactly three U-orbits of elements, of principal left ideals, of left ideals, and of principal right ideals.

$*$ If $n = 2$ (respectively, $n = 3$), then R has exactly four (five) U-orbits of right ideals.

$*$ If $n \geq 4$, then R has an infinite number of U-orbits of right ideals. $\square$

Example. Let $\mathbb{L}$ be a field, let $\mathbb{F} = \mathbb{L}(x_1, x_2, \ldots)$ be the field of rational functions in infinitely many variables $x_1, x_2, \ldots$, and let $\sigma \colon \mathbb{F} \to \mathbb{F}$ be the $\mathbb{L}$-endomorphism defined by $\sigma(x_i) = {x_i}^2$ for every $i \geq 1$. Let $R = \mathbb{F}[y; \sigma]/(y^2)$. Then

* R is left, but not right artinian.

* R has exactly three U-orbits of elements, of principal left ideals, of left ideals, and of principal right ideals.

* R has an infinite number of U-orbits of right ideals. □

Example. Let R be the same as in the previous example, let R^{op} be the opposite ring, and let $S = R \times R^{op}$. Then

* S is semiprimary, but neither left nor right noetherian.

* S has exactly nine U-orbits of elements, of principal left ideals, and of principal right ideals.

* S has an infinite number of U-orbits both left and right ideals. □

Example. Let $\mathbb{K}$ be an infinite subfield of a field $\mathbb{F}$, and let $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{K} \end{bmatrix}$ be the ring of matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $x, y \in \mathbb{F}$ and $z \in \mathbb{K}$, with formal matrix multiplication. Then

* R has exactly five U-orbits of elements, of principal left ideals, and of principal right ideals.

* R has exactly six U-orbits of left ideals.

* If $[\mathbb{F} : \mathbb{K}] = 1$, then R has exactly six U-orbits of right ideals.

* If $[\mathbb{F} : \mathbb{K}] = 2$ (respectively, $[\mathbb{F} : \mathbb{K}] = 3$), then R has exactly eight (ten) U-orbits of right ideals.

* If $[\mathbb{F} : \mathbb{K}] \geq 4$, then R has an infinite number of U-orbits of right ideals. $\square$

Example. Let $\mathbb{F}$ be an infinite field, let $D = diag(\mathbb{F}, \mathbb{F})$ be the 2×2 diagonal matrix ring, let $M = M_2(\mathbb{F})$ be the 2×2 matrix ring, and let $R = \begin{bmatrix} D & M \\ 0 & D \end{bmatrix}$. Then

* R is an 8-dimensional $\mathbb{F}$-algebra.

* R has a finite number of U-orbits of ideals.

* R has an infinite number of U-orbits of principal left ideals. $\square$